TRINITY COLLEGE

Semester Two Examination, 2022

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3&4

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	49	35
Section Two: Calculator-assumed	12	12	100	91	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (91 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(8 marks)**

The distribution of the weights W of loaves of rye bread produced by a bakery has a mean and standard deviation of 735 g and 15 g respectively. Quality control frequently take random samples of 60 white loaves from the bakery and calculate the mean weight of each sample.

(a) Describe the expected distribution of these sample means. (3 marks)

Solution
$s = 15 \div \sqrt{60} = 1.9365$ Sample means will be normally distributed with a mean of 735 g and standard deviation of 1.9365 g.
Specific behaviours
✓ states normally distributed ✓ states mean ✓ calculations standard deviation

Further production checks are made if the mean weight of a sample is less than a prescribed minimum value of 731.3 g.

(b) Over the course of the next 250 random samples, how many times would you expect that further production checks need to be made? (2 marks)

Solution
$P(\overline{W} < 731.3) = 0.0280$ Hence, expect to make further checks $250 \times 0.028 = 7$ times.
Specific behaviours
✓ calculates probability ✓ calculates expected number of times

Quality control has to reduce the sample size from 60 to 49 and change the prescribed minimum value so that the frequency of further production checks remains the same.

(c) Determine the prescribed minimum value for the mean weight of a sample required for this change. (3 marks)

Solution
New standard deviation of sampling distribution will be $s = 15 \div \sqrt{49} = 2.1429$. Required z-score for $p = 0.028$ is $z = -1.911$. Hence $\dfrac{w - 735}{2.1429} = -1.911 \Rightarrow w = 730.9$ The prescribed minimum value should be changed to 730.9 g.
Specific behaviours
✓ calculations new standard deviation ✓ obtains z-score for required probability ✓ calculates required value

Question 9 **(6 marks)**

The position vectors of points P and Q are $\begin{pmatrix}3\\0\end{pmatrix}$ and $\begin{pmatrix}11\\2\end{pmatrix}$ respectively.

(a) Determine the vector equation of line L that passes through P and Q. (1 mark)

Solution
$\overrightarrow{PQ} = \begin{pmatrix}11\\2\end{pmatrix} - \begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}8\\2\end{pmatrix} = 2\begin{pmatrix}4\\1\end{pmatrix}$ $\mathbf{r} = \begin{pmatrix}3\\0\end{pmatrix} + \mu\begin{pmatrix}4\\1\end{pmatrix}$
Specific behaviours
✓ correct vector equation

The vector equation of curve C is $\mathbf{r} = \begin{pmatrix}4\lambda^2 - 9\\2\lambda - 4\end{pmatrix}$.

(b) Determine the Cartesian equation of curve C. (2 marks)

Solution
$y = 2\lambda - 4 \rightarrow \lambda = \frac{y+4}{2}$ $x = 4\left(\frac{y+4}{2}\right)^2 - 9$ $x = y^2 + 8y + 7$
Specific behaviours
✓ expresses λ in terms of x or y ✓ obtains Cartesian equation (any form)

(c) Determine the position vector(s) of the point(s) where curve C meets line L. (3 marks)

Solution
Equating coefficients: $4\lambda^2 - 9 = 3 + 4\mu$ $2\lambda - 4 = \mu$ Solving simultaneously gives $\lambda = 1, \mu = -2$. $\begin{pmatrix}3\\0\end{pmatrix} - 2\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}-5\\-2\end{pmatrix}$ Hence position vector of point is $\begin{pmatrix}-5\\-2\end{pmatrix}$.
Specific behaviours
✓ forms simultaneous equations   ✓ solves equations for λ and/or μ ✓ substitutes to obtain correct position vector

Question 10 **(9 marks)**

Let $f(x) = \ln(5 - \sqrt{x+4})$ and $g(x) = x^2 - 4x$.

(a) State the domain and range of f. (3 marks)

Solution
For natural domain we require $5 - \sqrt{x+4} > 0$. Hence $x + 4 \geq 0 \Rightarrow x \geq -4$ and $\sqrt{x+4} < 5 \Rightarrow x < 21$. $D_f = \{x \mid x \in \mathbb{R}, -4 \leq x < 21\}$ Range will be all values of $\ln k$ where $0 < k \leq 5$. Hence $R_f = \{y \mid y \in \mathbb{R}, y \leq \ln 5\}$
Specific behaviours
✓ indicates at least one required condition / inequality ✓ correct domain ✓ correct range

(b) Determine $f^{-1}(x)$ and state its range. (3 marks)

Solution
$x = \ln(5 - \sqrt{y+4})$ $e^x = 5 - \sqrt{y+4}$ $y = (5 - e^x)^2 - 4$ $f^{-1}(x) = e^{2x} - 10e^x + 21$ $R_{f^{-1}} = D_f = \{x \mid x \in \mathbb{R}, -4 \leq x < 21\}$
Specific behaviours
✓ indicates appropriate steps to obtain inverse ✓ correct inverse (factored or expanded form) ✓ correct range

(c) Determine an expression for $f \circ g(x)$ and state the domain for which the composite function is defined. (3 marks)

Solution
$f \circ g(x) = \ln\left(5 - \sqrt{x^2 - 4x + 4}\right)$ $= \ln(5 - \lvert x - 2 \rvert)$ For the domain we require $5 - \lvert x - 2 \rvert > 0 \Rightarrow -3 < x < 7$. $D_{f \circ g} = \{x \mid x \in \mathbb{R}, -3 < x < 7\}$
Specific behaviours
✓ correct expression for composite function ✓ indicates required condition ✓ correct domain

Question 11 **(6 marks)**

The position vector of a particle moving in the Cartesian plane at time t seconds is given by

$$\mathbf{r}(t) = 3\cos(4t)\,\mathbf{i} - 3\sin(4t)\,\mathbf{j}.$$

(a) Show that the particle is moving at a constant speed. (2 marks)

Solution
Velocity: $$\mathbf{v}(t) = \frac{d}{dt}\big(\mathbf{r}(t)\big) = -12\sin(4t)\,\mathbf{i} - 12\cos(4t)\,\mathbf{j}.$$ Speed: $$\begin{aligned}|\mathbf{v}(t)| &= 12\sqrt{(-\sin(4t))^2 + (-\cos(4t))^2} \\ &= 12 \times 1 \\ &= 12\end{aligned}$$
Specific behaviours
✓ differentiates to obtain velocity vector ✓ uses Pythagorean identity to show speed is constant

(b) Calculate the scalar product of the position vector and the velocity vector of the particle and interpret the result. (2 marks)

Solution
$$\begin{aligned}\mathbf{r}(t)\cdot\mathbf{v}(t) &= \begin{pmatrix}3\cos(4t) \\ -3\sin(4t)\end{pmatrix}\cdot\begin{pmatrix}-12\sin(4t) \\ -12\cos(4t)\end{pmatrix} \\ &= -36\cos(4t)\sin(4t) + 36\cos(4t)\sin(4t) \\ &= 0\end{aligned}$$ Hence the position vector and the velocity vector of the particle are always perpendicular.
Specific behaviours
✓ calculates scalar product ✓ interprets result of scalar product

(c) Determine the acceleration vector of the particle when its position vector is $-3\mathbf{i}$. (2 marks)

Solution
$$\begin{aligned}\mathbf{a}(t) &= \frac{d}{dt}\big(\mathbf{v}(t)\big) \\ &= -48\cos(4t)\,\mathbf{i} + 48\sin(4t)\,\mathbf{j} \\ &= -16\mathbf{r}(t)\end{aligned}$$ Hence $\mathbf{a} = -16 \times -3\mathbf{i} = 48\mathbf{i}$.
Specific behaviours
✓ differentiates to obtain acceleration vector ✓ correct acceleration vector at given position

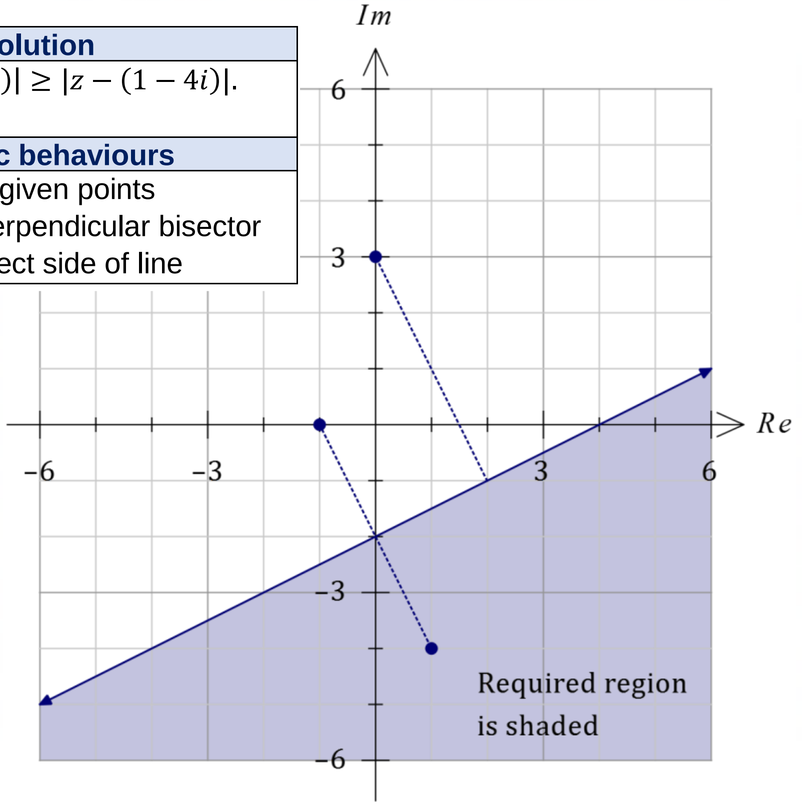
Question 12 **(8 marks)**

Consider the complex number z.

(a) Let R be the subset of the complex plane that satisfies $|z+1| \geq |z-1+4i|$.

(i) Sketch the subset R. (3 marks)

Solution
R: $\lvert z-(-1+0i)\rvert \geq \lvert z-(1-4i)\rvert$. See diagram.
Specific behaviours
✓ indicates two given points ✓ draws their perpendicular bisector ✓ indicates correct side of line

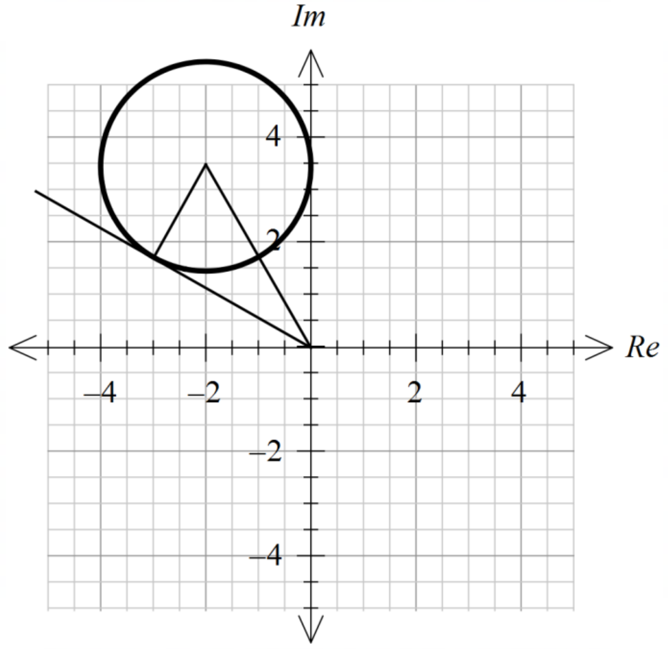


(ii) Determine the exact minimum value of $|z-3i|$ in R. (2 marks)

Solution
Require minimum distance from R to $3i$. $d = 2\sqrt{1^2+2^2} = 2\sqrt{5}$ units
Specific behaviours
✓ indicates value as distance ✓ correctly calculates distance

(b) Determine the greatest value of $\arg(z)$ if $|z + 2 - 2\sqrt{3}i| \leq 2$. (3 marks)

Solution



Let θ be the angle of the centre of the circle. Then

$$\tan\theta = \frac{2\sqrt{3}}{2}$$
$$\theta = \frac{\pi}{3}$$

Let α be the angle the tangent makes. Then

$$\sin\alpha = \frac{2}{4}$$
$$\alpha = \frac{\pi}{6}$$

$\therefore$ greatest value is

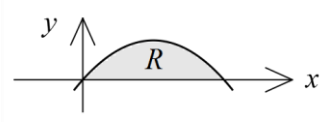
$$\arg(z) = \pi - \frac{\pi}{3} + \frac{\pi}{6}$$
$$= \frac{5\pi}{6}$$

Specific behaviours

✓ circle
✓ correct rays in complex plane
✓ correct angle

Question 13 **(8 marks)**

Let $f(x) = x(k - x)$, where k is a constant, and region R be the area between the x-axis and the curve $y = f(x)$.



All dimensions are in centimetres.

(a) When $k = 12$, determine the volume of revolution when R is rotated about the y-axis. (5 marks)

Solution

Expressions for inner and outer curves:

$$y = x(12 - x) \Rightarrow x = 6 \pm \sqrt{36 - y}$$

When $x = 12 \div 2,\ y_{max} = 36.$

$$V_O = \pi \int_0^{36} \left(6 + \sqrt{36 - y}\right)^2 dy \quad (= 3672\pi \approx 11536)$$

$$V_I = \pi \int_0^{36} \left(6 - \sqrt{36 - y}\right)^2 dy \quad (= 216\pi \approx 679)$$

$$\begin{aligned} V &= V_0 - V_I \\ &= \pi \int_0^{36} 24\sqrt{36 - y}\, dy \\ &= 3456\pi \approx 10\,857 \text{ cm}^3 \end{aligned}$$

Specific behaviours

✓ expressions for inner and outer curves
✓ obtains maximum value of y
✓ indicates correct definite integral for inner or outer volume
✓ indicates appropriate method to obtain required volume
✓ obtains correct volume

(b) When R is rotated about the x-axis, the volume of revolution is $\dfrac{512\pi}{15}$ cm³.
Determine the value of k. (3 marks)

Solution

$$\begin{aligned} V &= \pi \int_0^{k} \left(x(k - x)\right)^2\, dx \\ &= \frac{k^5\pi}{30} \end{aligned}$$

$$\frac{k^5\pi}{30} = \frac{512\pi}{15}$$
$$k = 4$$

Specific behaviours

✓ correct definite integral for volume
✓ obtains expression for volume in terms of k
✓ forms equation and solves for k

Question 14 **(9 marks)**

The mean and standard deviation of a random sample of 47 art teachers working in a region was 42.6 and 7.1 years respectively. The sample was taken to construct a confidence interval for the mean age of art teachers.

(a) State two reasons why it is appropriate to assume the approximate normality of the distribution of the sample mean for this data. (2 marks)

Solution
Sampling is random and sample size of 47 is large (i.e., exceeds 30).
Specific behaviours
✓ states sampling is random ✓ states sample size is large

(b) State another assumption required to construct a valid confidence interval. (1 mark)

Solution
- sample standard deviation is a good estimate for the population standard deviation. - sample values are independent of each other
Specific behaviours
✓ states one valid assumption

(c) Construct a 95% confidence interval for the mean age of art teachers working in the region. (3 marks)

Solution

$$s = 7.1 \div \sqrt{47} = 1.036, \qquad z_{0.95} = 1.96$$

Interval:

$$42.6 - 1.96(1.036) \le \mu \le 42.6 + 1.96(1.036)$$
$$42.6 - 2.03 \le \mu \le 42.6 + 2.03$$

$$40.57 \le \mu \le 44.63$$

Specific behaviours

✓ standard deviation of sampling distribution
✓ correct expression for confidence interval
✓ correct confidence interval

(d) Based on another random sample, the 90% confidence interval for the mean age of sports teachers employed in the same region was calculated to be $(37.89, 40.91)$. Given that the standard deviation of the sample was 7.4 years, determine the size of the sample. (3 marks)

Solution

$$z_{0.90} = 1.645$$

$$E = (40.91 - 37.89) \div 2 = 1.51$$

$$n = \left(\frac{1.645 \times 7.4}{1.51}\right)^2 = 65$$

Specific behaviours

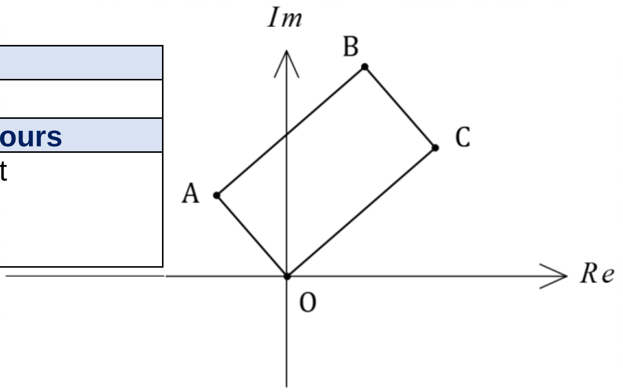
✓ margin of error E
✓ correct equation for sample size
✓ correct sample size n

Question 15 **(7 marks)**

Let $OABC$ be a rectangle in the complex plane, where O is the origin. The points A and C represent the complex numbers z and $-\sqrt{3}iz$ respectively, where $Re(z) < 0$ and $Im(z) > 0$.

(a) Draw a labelled sketch of the rectangle in the complex plane. (3 marks)

Solution
See diagram
Specific behaviours
✓ A in second quadrant ✓ $\angle AOC \approx 90°$ ✓ C in first quadrant



(b) Determine the complex number represented by B. (2 mark)

Solution
$B = z + (-\sqrt{3}i)z$ $= (1 - \sqrt{3}i)z$
Specific behaviours
✓ addition of complex numbers ✓ correct complex number for B

Rectangle $OABC$ is rotated $120°$ about O in an anticlockwise direction to $OA'B'C'$.

(c) Determine in exact Cartesian form the complex numbers represented by the point B'. (2 marks)

Solution
Let $w = \text{cis}(120°) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ so that multiplying any complex number by w will rotate it $120°$ anticlockwise in the complex plane. Then $B' = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(1 - \sqrt{3}i)z = (1 + \sqrt{3}i)z$
Specific behaviours
✓ indicates use of multiplication by complex number for rotation ✓ simplified complex number for B'

Question 16 **(8 marks)**

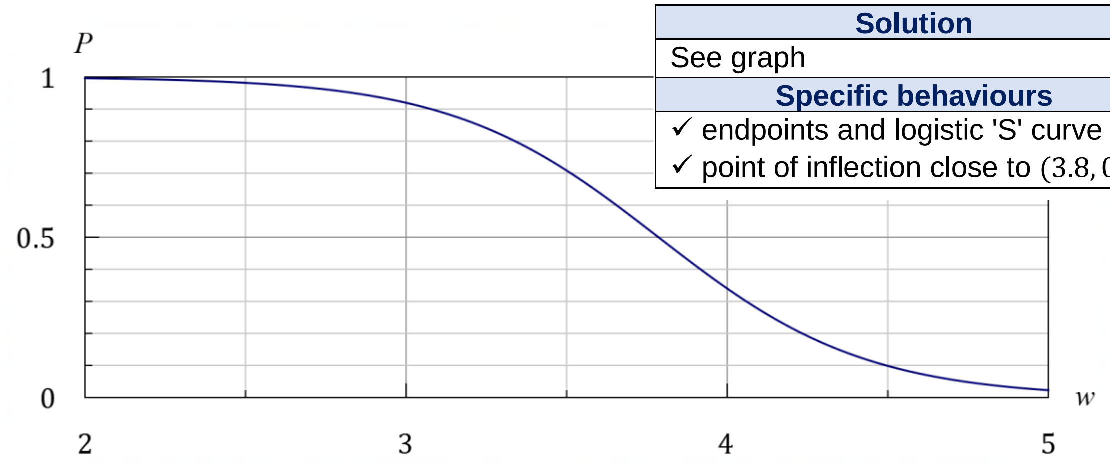
The probability P that an adult Siamese cat of weight w kg is female can be calculated using the logistic equation

$$P = \frac{1}{1 + 0.000\,008e^{3.1w}}.$$

(a) Calculate the probability that a cat of weight 3.2 kg is a female. (1 mark)

Solution
$P = 0.860$
Specific behaviours
✓ correct value

(b) Sketch the graph of this model on the axes below. (2 marks)



Solution
See graph
Specific behaviours
✓ endpoints and logistic 'S' curve
✓ point of inflection close to $(3.8, 0.5)$

(c) The logistic equation can be written in the form $\frac{dP}{dw} = rP(k - P)$. State the value of r and the value of k. (2 marks)

Solution
k is limiting value as $w \to -\infty$ and so $k = 1$. From defining rule, $-rk = 3.1 \Rightarrow r = -3.1$.
Specific behaviours
✓ value of k
✓ value of r

(d) The sensitivity S of the model is defined as the absolute value of the change in P for a one-gram increase in the weight of a cat. Determine the maximum value of S. (3 marks)

Solution
P changing fastest at point of inflection, when $P = 0.5$. $\frac{dP}{dw} = -3.1(0.5)(1 - 0.5) = -0.775$ units/kg This rate is in kg, hence $S_{MAX} = 0.775 \div 1000 = 0.000\,775$ u/g.
Specific behaviours
✓ indicates where the rate of change is greatest
✓ obtains value of $\frac{dp}{dw}$ at point of inflection
✓ correct maximum value of S

Question 17 **(8 marks)**

Plane Π contains triangle OAB.

Relative to O, the points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$ respectively.

(a) State the unit vectors $\hat{\mathbf{a}}$ and $\hat{\boldsymbol{b}}$. (1 mark)

Solution
$\hat{\mathbf{a}} = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}, \quad \hat{\mathbf{b}} = \frac{1}{7}\begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$
Specific behaviours
✓ correct unit vectors

(b) Calculate $\angle AOB$. (1 mark)

Solution
$\angle AOB$ is the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.
Using CAS, $\angle AOB = \cos^{-1}\left(\frac{8}{21}\right) = 67.6°$.
Specific behaviours
✓ correct angle

(c) Determine the equation of plane Π in the form $\mathbf{r} \cdot \mathbf{n} = k$. (2 marks)

Solution
$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{pmatrix} -7 \\ -18 \\ 2 \end{pmatrix}$
Hence equation of plane is $\mathbf{r} \cdot \begin{pmatrix} 7 \\ 18 \\ -2 \end{pmatrix} = 0$.
Specific behaviours
✓ indicates correct normal to plane
✓ correct equation of plane (any multiple of $\mathbf{n}$)

Point C with position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ lies in plane Π and within triangle OAB so that $|\overrightarrow{OC}| = 1$ and OC bisects $\angle AOB$.

(d) Explain why the values of x, y and z must satisfy the equation $7x + 18y - 2z = 0$. (1 mark)

Solution
For C to lie in plane, it must satisfy the equation for plane Π:
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 18 \\ -2 \end{pmatrix} = 0 \Rightarrow 7x + 18y - 2z = 0$
Specific behaviours
✓ explains that C lies in plane and so must satisfy equation from (c)

(e) Determine two other equations that the values of x, y and z must satisfy and hence, or otherwise, determine vector $\overrightarrow{OC}$, giving components rounded to three decimal places.

(3 marks)

Solution

So that $|\overrightarrow{OC}| = 1$ (note that $\overrightarrow{OC}$ is a unit vector):

$$\left|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right| = 1 \Rightarrow x^2 + y^2 + z^2 = 1 \quad \ldots (1)$$

To have $\angle AOC = \angle COB$, then $\hat{\mathbf{a}} \cdot \overrightarrow{OC} = \overrightarrow{OC} \cdot \hat{\mathbf{b}}$:

$$\frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{1}{3}(2x - y - 2z) = \frac{1}{7}(6x - 2y + 3z)$$

$$4x + y + 23z = 0 \quad \ldots (2)$$

Solving equations $(1), (2)$ and from (d) simultaneously using CAS gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \pm\frac{\sqrt{1218}}{1218}\begin{pmatrix} -32 \\ 13 \\ 5 \end{pmatrix} \approx \pm\begin{pmatrix} -0.917 \\ 0.372 \\ 0.143 \end{pmatrix}$$

So that C lies in ΔOAB, $x > 0$ and so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{\sqrt{1218}}{1218}\begin{pmatrix} 32 \\ -13 \\ -5 \end{pmatrix} \approx \begin{pmatrix} 0.917 \\ -0.372 \\ -0.143 \end{pmatrix}$$

NB Other methods exist to determine $\overrightarrow{OC}$, such as finding point P where the angle bisector intersects AB and thus obtaining the required unit vector. Beware of erroneous arguments such as the angle bisector will bisect side AB, etc., but otherwise award one mark for an alternative approach that results in the correct vector. Note that $P(3.2, -1.3, -0.5)$.

Specific behaviours

✓ forms equation using magnitude of $\hat{\mathbf{c}}$
✓ forms equation so that $\overrightarrow{OC}$ is bisector (no simplification required)
✓ obtains unit vector as required (3 d.p. for guidance only)

Question 18 **(7 marks)**

(a) Let the complex numbers $z = r \operatorname{cis}\theta$ and $w = \dfrac{-1+i}{2z}$. Determine the modulus and argument of w in terms of the real constants r and θ.

(2 marks)

Solution
$w = \dfrac{-1+i}{2z} = \dfrac{\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)}{2r\operatorname{cis}\theta} = \dfrac{\sqrt{2}}{2r}\operatorname{cis}\left(\dfrac{3\pi}{4} - \theta\right)$ Hence $\lvert w\rvert = \dfrac{\sqrt{2}}{2r}$ and $\arg(w) = \dfrac{3\pi}{4} - \theta$.
Specific behaviours
✓ modulus   ✓ argument

(b) Let the complex numbers $z = a + 2i$ and $w = \dfrac{z-3i}{z+i}$. Determine the value(s) of the real constant a given that w is purely imaginary. (2 marks)

Solution
$w = \dfrac{a-i}{a+3i} = \dfrac{a^2-3}{a^2+9} - \dfrac{4a}{a^2+9}i$ $Re(w) = 0 \Rightarrow a^2 - 3 = 0 \Rightarrow a = \pm\sqrt{3}$.
Specific behaviours
✓ expresses w in real and imaginary parts   ✓ correct values of a

(c) The complex number $w = 5\operatorname{cis}\left(-\dfrac{6\pi}{13}\right)$ is a root of the equation $z^n - (a+bi) = 0$, where n, a and b are real non-zero constants.

Determine two roots of the complex equation $z^{2n} - (a+bi) = 0$. (3 marks)

Solution
$z^{2n} = (z^2)^n = a + bi \Rightarrow z^2 = w$ is a solution, so require square roots of $5\operatorname{cis}\left(-\dfrac{6\pi}{13}\right)$. $z_n = \sqrt{5}\operatorname{cis}\left(-\dfrac{6\pi}{13\times 2} + \dfrac{2n\pi}{2}\right), \quad n = 0, 1$ $z_0 = \sqrt{5}\operatorname{cis}\left(-\dfrac{3\pi}{13}\right), \quad z_1 = \sqrt{5}\operatorname{cis}\left(\dfrac{10\pi}{13}\right)$
Specific behaviours
✓ indicates square roots of w required   ✓ one root   ✓ second root

Question 19 **(7 marks)**

A small body is moving in a straight line so that t seconds after leaving fixed point O its velocity is v cm/s and its acceleration $a = bv + c$ cm/s^2, where b and c are constants.

Initially the body is at rest at O and its acceleration is 3.6 cm/s^2.

T seconds later, its velocity is 4 cm/s and its acceleration is 0.4 cm/s^2.

(a) Show that $5\dfrac{dv}{dt} + 4v - 18 = 0$. (3 marks)

Solution
When $t = 0, v = 0, a = 3.6$ and when $t = T, v = 5, a = -0.4$. Using $a = bv + c$: $3.6 = b(0) + c \Rightarrow c = 3.6$ $0.4 = 4b + 3.6 \Rightarrow b = -0.8$ Hence $a = -0.8v + 3.6$ $5a = -4v + 18$ $5\dfrac{dv}{dt} + 4v - 18 = 0$
Specific behaviours
✓ obtains value of c ✓ obtains value of b ✓ uses $a = \frac{dv}{dt}$ to obtain equation

(b) Determine the exact value of T. (4 marks)

Solution
$\dfrac{dv}{dt} = \dfrac{18 - 4v}{5}$ $\displaystyle\int \frac{dv}{18 - 4v} = \int \frac{dt}{5}$ $-\dfrac{1}{4}\ln\lvert 18 - 4v\rvert = \dfrac{t}{5} + C$ When $t = 0, v = 0$ and so $C = -\dfrac{1}{4}\ln 18$ When $t = T, v = 4$ and so $\dfrac{T}{5} = \dfrac{1}{4}\ln 18 - \dfrac{1}{4}\ln 2$ $T = \dfrac{5}{4}\ln 9$ $= \dfrac{5}{2}\ln 3\text{ s} \quad (\approx 2.75\text{ s})$
Specific behaviours
✓ separates variables   ✓ obtains correct antiderivative   ✓ evaluates constant of integration   ✓ obtains exact value of T

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________